A New Bispectral Test for NonLinear Serial Dependence

Elena Rusticelli*, Richard A. Ashley, Estela Bee Dagum***

and Douglas M. Patterson***

Abstract. Nonconstancy of the bispectrum of a time series has been taken as a measure of non-Gaussianity and nonlinear serial dependence in a stochastic process by Subba Rao and Gabr (1980) and by Hinich (1982), leading to Hinich's statistical test of the null hypothesis of a linear generating mechanism for a time series. Hinich's test has the advantage of focusing directly on nonlinear serial dependence — in contrast to subsequent approaches, which actually test for serial dependence of any kind (nonlinear or linear) on data which have been pre-whitened. The Hinich test tends to have low power, however, and (in common with most statistical procedures in the frequency domain) requires the specification of a smoothing or window-width parameter. In this paper we develop a modification of the Hinich bispectral test which substantially ameliorates both of these problems by the simple expedient of maximizing the test statistic over the feasible values of the smoothing parameter. Monte Carlo simulation results are presented indicating that the new test is well-sized and has substantially larger power than the original Hinich test against a number of relevant alternatives; the simulations also indicate that the new test preserves the Hinich test's robustness to mis-specifications in the identification of a pre-whitening model.

Key Words. Bispectrum, smoothing parameter, maximization technique, nonlinearity, linear prefiltering procedure.

* Department of Statistics, University of Bologna, Bologna, Italy.

** Department of Economics, Virginia Polytechnic Institute and State University, Blacksburg, USA.

*** Department of Finance, Virginia Polytechnic Institute and State University, Blacksburg, USA.

1 INTRODUCTION

The introduction of frequency domain tests for nonlinear serial dependence increased the use of bispectral analyses on dependent data. Initially, Subba Rao and Gabr (1980) implemented Brillinger's (1965) method for measuring the departure of a process from linearity and Gaussianity by estimation of the bispectrum of observed data. These authors however did not consider the asymptotic sampling properties of the bispectrum, which were developed by Rosenblatt and Van Ness (1965), Shaman (1965), and Brillinger and Rosenblatt (1967). In 1982, Hinich proposed a nonparametric bispectral procedure to test the null hypothesis of linearity and Gaussianity, obtaining a chi-squared statistic for testing the significance of individual bispectrum estimates by exploiting its asymptotic distribution.

An important advantage of bispectral analysis is its invariance with respect to linear filtering of the original sample data. Indeed, the common practice of linearly prefiltering the observed data set, in order to remove possible autocorrelation and reveal the presence of nonlinear dependence hidden by some dominant type of linear dependence, can lead to both mis-specified nonlinear models and distorted statistical inferences if the prefiltering procedure is not correctly applied. For this reason, a statistical tool which is robust to possible mis-specification in the prefiltering linear model is highly advantageous.

When a stochastic process $\{X_t\}$ is Gaussian then all its polyspectra (spectra of order higher than the second) are identically zero. Consequently, if a process presents a non-zero bispectrum this could be due to two reasons: the process $\{X_t\}$ conforms to a linear model but the innovations $\{\varepsilon_t\}$ are non-Gaussian, or $\{X_t\}$ conforms to a non-linear model with $\{\varepsilon_t\}$ being either Normal or non-Normal.

Subba Rao and Gabr firstly, and Hinich later, considered specific features in the bispectrum shape of linear and Gaussian stochastic processes to construct two different tests: one for the hypothesis of linearity and the other for Gaussianity. In this study the attention will be focused only on the bispectral test for linearity because a rejection of the null hypothesis of linearity leads automatically to a rejection of Gaussianity.

This paper is organized as follows. Section 2 describes the Hinich bispectral test for linearity, together with its main limitations regarding the role of the smoothing parameter in determining the test statistic consistency. Section 3 describes a set of new bispectral tests for linearity, in which a maximization procedure is applied so as to both eliminate the arbitrariness concerning the smoothing parameter and to increase the test's diagnostic power. Section 4 presents a Monte Carlo simulation study where the reliability of the new tests is evaluated in terms of size and in terms of power with respect to several nonlinear alternatives. The invariance property is verified in section 5, where the sizes of the tests for data which have been prefiltered using a mis-specified linear model are examined.

2 THE HINICH BISPECTRAL TEST FOR LINEARITY

Let $\{X_t\}$ be a third-order stationary stochastic process for which $E[X_t] = 0$ for all t. The double Fourier transformation of its third-order cumulant function at the frequency pair (f_1, f_2), the *bispectrum*, is given by

$$B_X(f_1, f_2) = \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} \gamma_{m,n} e^{-i2\pi(f_1 m + f_2 n)}, \tag{1}$$

where the symmetry of the bicovariance function, $\gamma_{m,n} = E[X_t, X_{t-m}, X_{t-n}]$, implies a triangular principal domain for the bispectrum, i.e. $D = \{0 < f_1 < \frac{1}{2},\ f_2 < f_1,\ 2f_1 + f_2 < 1\}$. An analytical treatment of the bispectrum is provided by Brillinger and Rosenblatt (1967).

When $\{X_t\}$ is a linear process, such that

$$X_t = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}, \tag{2}$$

where $\{\varepsilon_t\}$ is $IID(0, \sigma_\varepsilon^2)$, and $\sum_{j=0}^{\infty} |\psi_j| < \infty$, then the bispectrum is given by

$$B_X(f_1, f_2) = \mu_3 \Psi(f_1) \Psi(f_2) \Psi^*(f_1 + f_2), \tag{3}$$

where $\mu_3 = E[\varepsilon_t^3]$ is the third-order moment of the innovation process, $\Psi(f) = \sum_{j=0}^{\infty} \psi_j e^{-i2\pi f j}$ $= \Psi(e^{-i2\pi f})$ the transfer function of the filter, and $\Psi^*(f) = \Psi(-f)$ its complex conjugate.

Denoting the power spectrum of $\{X_t\}$ by

$$S_X(f) = \sigma_\varepsilon^2 |\Psi(f)|^2, \tag{4}$$

it follows that

$$\frac{\left|B_X\left(f_1,f_2\right)\right|^2}{S_X\left(f_1\right)S_X\left(f_2\right)S_X\left(f_1+f_2\right)}=\frac{\mu_3^2}{\sigma_\varepsilon^6}=\Gamma_X^2(f_1,f_2), \tag{5}$$

where $\Gamma_X^2(f_1,f_2)$ is the squared Fisher's *skewness function*. Hence, if the stochastic process $\{X_t\}$ is linear, then the skewness function $\Gamma_X\left(f_1,f_2\right)=\mu_3/\sigma_\varepsilon^3$ is constant for all frequency pairs (f_1,f_2) in D, i.e. independent of both time and frequency pair (f_1,f_2). Furthermore, if $\{X_t\}$ is also Gaussian, then $\mu_3=0$ and consequently $\Gamma_x\left(f_1,f_2\right)=0$ for all frequency pairs.

These two properties of the bispectrum of a linear process have been exploited by Subba Rao and Gabr (1980) and by Hinich (1982) to test the null hypothesis of linearity and of Gaussianity.

A consistent estimator of the bispectrum at frequency pair (f_m,f_n) is obtained by smoothing the third-order periodogram $F_X\left(j,k\right)$ over adjacent frequency pairs as

$$\widehat{B}_X\left(f_m,f_n\right)=\frac{1}{M^2}\sum_{j=(m-1)M}^{mM-1}\;\sum_{k=(n-1)M}^{nM-1}F_X\left(j,k\right), \tag{6}$$

where $F_X\left(j,k\right)=X\left(f_j\right)X\left(f_k\right)X^*\left(f_{j+k}\right)$ and $X\left(f_j\right)=\sum_{t=0}^{N-1}x_te^{-i2\pi f_jt}$ is the Fourier transformation of the sample series $\{x_1,x_2,...,x_N\}$ from the process $\{X_t\}$. $\widehat{B}_X\left(f_m,f_n\right)$ is the expected value of $F_X\left(j,k\right)$ over a lattice square L of M^2 points, whose points outside the principal domain D are not included in the average.

Hinich (1982) showed that the estimator $2\left|\widehat{X}\left(f_m,f_n\right)\right|^2$, referred as *estimated standardized bispectrum*, is asymptotically a non central chi-square distribution with two degrees of freedom, $\chi_2^2\left(\lambda_{m,n}\right)$, where

$$\widehat{X}\left(f_m,f_n\right)=\frac{\widehat{B}_X\left(f_m,f_n\right)}{\sqrt{\frac{NQ_{m,n}}{M^4}\left[\widehat{S}_X\left(f_m\right)\widehat{S}_X\left(f_n\right)\widehat{S}_X\left(f_{m+n}\right)\right]}} \tag{7}$$

is asymptotically a standard normal and $\widehat{S}_X\left(.\right)$ is a consistent estimator of the power spectrum and $f_j=(2j-1)M/2N$ for any integer j. The noncentrality parameter $\lambda_{m,n}$ is given by

$$\lambda_{m,n}=2\left(NQ_{m,n}/M^4\right)^{-1}\frac{\left|B_X\left(f_m,f_n\right)\right|^2}{S_X\left(f_m\right)S_X\left(f_n\right)S_X\left(f_{m+n}\right)} \tag{8}$$

where $Q_{m,n}$ is equal to the number of all pairs (j,k) within the lattice square inside the triangular principal domain, but not on the boundaries of the triangle $j = k$, or $2j + k = N$. If the whole square is inside the domain, then $Q_{m,n} = M^2$, otherwise $Q_{m,n} < M^2$. Hence, the value of the noncentrality parameter increases when a smaller set of frequency pairs (m, n) is considered, i.e.

$$\lambda_{m,n} = 2\left(NQ_{m,n}/M^4\right)^{-1}\Gamma_X^2(f_m, f_n) \geq 2\left(N/M^2\right)^{-1}\Gamma_X^2(f_m, f_n). \tag{9}$$

The Hinich bispectral test considers the distribution of the estimated standardized bispectrum $2\left|\widehat{X}\left(f_m, f_n\right)\right|^2$ in order to evaluate the null hypothesis of linearity. In fact, when the process $\{X_t\}$ is linear, the parameter $\lambda_{m,n}$ is a constant (λ_0) for all P pairs (m, n) considered in the principal domain. Consequently, the estimates of the standardized bispectrum constitute P realizations, all drawn from the same noncentral chi-square distribution $\chi_2^2(\lambda_0)$. Hinich (1982) and Ashley, Patterson and Hinich (1986) then calculate a robust, non-parametric, dispersion measure on the sample distribution of $2\left|\widehat{X}\left(f_m, f_n\right)\right|^2$ and reject the hypothesis of constancy in the standardized bispectrum when this sample dispersion is too large. In particular, the interquartile range IQR, the interdecile range IDR, and the 80% fractile are used to measure this dispersion; the statistical significance of these measures is evaluated using their asymptotic distributions, as given by David (1970).

2.1 CONSISTENCY OF THE BISPECTRUM ESTIMATOR AND THE CLASSICAL HINICH TEST

The aforementioned procedure for smoothing the bispectrum estimator $F_X(j, k)$, over a square of M^2 possible points centered within a lattice square L in the principal domain, reduces the estimator's sampling variance but introduces bias.

Letting M equal the integer part of N^c, Hinich (1982) showed that c must lie in the range $(1/2, 1)$ in order for $\widehat{B}_x\left(f_m, f_n\right)$ to be a consistent estimator of $B_x\left(f_m, f_n\right)$. Indeed, the variance of $\widehat{B}_x\left(f_m, f_n\right)$ is given by

$$Var\left[\widehat{B}_X\left(f_m, f_n\right)\right] = \frac{NQ_{m,n}}{M4}\left[\widehat{S}_X\left(f_m\right)\widehat{S}_X\left(f_n\right)\widehat{S}_X\left(f_{m+n}\right)\right] + O\left(M/N\right), \tag{10}$$

with $f_j = (2j-1)M/2N$ for any integer j. $M/N \rightarrow 0$ if $\frac{1}{2} < c < 1$ and $N \rightarrow \infty$, whereas $NM^{-4}Q_{m,n} \leq NM^{-2} = N^{1-2c} \rightarrow 0$ if $N \rightarrow \infty$, since $Q_{m,n} \leq M^2$.

The smallest sampling variance is obtained for $c = 1$, whereas the smallest bias is for $c = 1/2$. When $c < 1/2$, the bispectrum estimator is no longer consistent (and there is a large number of terms to sort out for the linearity test), so the sampling variance can be overly large in that case. Conversely, when c is much larger than $1/2$, then the bandwith is large, the sampling variance is reduced, and the estimator is relatively precise. However, in this latter case P — the total number of frequency pairs whose lattice square center lies in the principal domain — is small, yielding only a few estimable standardized bispectrum terms available with which to estimate the dispersion of the bispectrum across the frequency pairs.

When the parameter c is set to a value outside the range $\frac{1}{2} < c < 1$, the asymptotic convergence to the standard normal distribution for the bispectrum estimator, $\widehat{X}\left(f_{m,}f_{n}\right)$ given in equation (7), and consequently the convergence to the noncentral chi-square distribution for the estimated standardized bispectrum, $2\left|\widehat{X}\left(f_{m,}f_{n}\right)\right|^{2}$, becomes uncertain. Ashley, Patterson and Hinich (1986) empirically demonstrated that a larger value of M yields a slower convergence rate for the linearity test.

Brock, Hsieh and LeBaron (1991) and others have demonstrated that the accuracy of the asymptotic approximations for the sampling distributions of a number of frequently considered nonlinarity tests is inadequate for reasonable sample sizes. The asymptotic convergence to the limiting distribution for many nonlinearity tests can also be seriously compromised where the underlying moments condition fail (e.g., De Lima, 1997). For these reasons the rejection significance levels for the Hinich test (in common with many other nonlinearity tests) are in practice evaluated using nonparametric bootstrap methods.

3 A NEW NONLINEARITY TEST BASED ON THE HINICH BISPECTRAL TEST

In this section we propose a new nonlinearity test which is a modification of the classical Hinich bispectral test.

Our new test eliminates the arbitrary selection of a smoothing parameter — M or c — by maximizing the test statistic over the feasible values of M. The problem of choosing the smoothing parameter M has been repeatedly discussed in the literature over the last two decades. A number of attempts have been made to obtain faster convergence to the asymptotic distribution by means of improved methods for smoothing the bispectral estimates. These studies have typically considered the size and power of both Hinich tests (the test against Gaussianity as well and the nonlinearity test), using generated data from different linear and nonlinear models. As noted above, Hinich (1982) showed that consistency requires M to be an integer exceeding $N^{0.5}$; Ashley, Patterson and Hinich (1986) examined the consequences of setting the smoothing constant equal to 30 percent less than $1 + N^{0.5}$. The rate of convergence for the linearity test was higher than that obtained with $M = 1 + N^{0.5}$, but this percentage reduction seriously compromised the reliability of the Gaussianity test. Lemos and Stokes (2000) considered averaging the interquartile range test statistic (Hinich, 1982) over M values ranging from $N^{0.5}$ to $(N/3)^{0.5}$. In none of these studies was it possible to identify any value or expression for M which was clearly superior. Consequently, Ashley and Patterson (2000) simply set M equal to the smallest integer less than $N^{0.6}$.

Another major concern is to make the test as much powerful as possible respect to reasonable alternative models for the generating process of the time series. This is a particularly important issue for the Hinich test because a number of simulation studies have shown that the it is not as powerful as some competing tests. In part this is because the test can be expected to have relatively small power against any forms of nonlinearity which cause flatness in the bispectrum of the stochastic process (e.g. Tong, 1990; Granger and Teräsvirta, 1993). In addition, simulation evidence indicates that the bispectral test has low power against forms of chaos that produce irregular and widely spaced spikes in the bispectrum, despite the strong nonlinearity which characterizes the generating mechanism (Ashley and Patterson, 1989; Barnett et al., 1997).

The maximization strategy proposed here elegantly addresses the arbitrariness in the choice of M. The empirical power results obtained in the simulation results reported below indicate that it also addresses the power concerns noted above. We note that several other studies (notably Andrews, 1989 and 1993; Bai, 1997; Bai and Perron, 1998 and 2003) have also used maximal values in constructing a diagnostic test statistic for regression parameter instability. Along the same lines, Cristiano (1992) discusses a maximal test statistic to detect trend breaks in GNP data; and White (2000) has proposed a maximization procedure, which he terms a "reality check", as a predictive model selection criteria.

If the hypothesis of linearity is not rejected for the value of M yielding the maximal dispersion in the sample standardized bispectrum, which implies the greatest discrepancy with respect to the hypothesis of flatness of the bispectrum, then it will not be rejected for any other value of M. Thus, using the maximal test statistic can be expected to increase the sensitivity (power) of the test with respect to extreme bispectrum values which appear as sharp, isolated peaks. Moreover, the robustness of the proposed test is enhanced by considering the entire range of computationally feasible values for the smoothing parameter range. Most importantly, however, by using the maximal value of the Hinich test statistic (i.e., of the sample dispersion of the estimated standardized bispectrum) across all feasible values of M, we eliminate the arbitrary choice of a value for this parameter.

3.1 THE MAXIMAL TEST STATISTICS

The test proposed here considers the maximal values of the sample dispersion statistics over the computationally feasible range of values for M. As in the classical Hinich test, the non-constancy of the standardized bispectrum $2\left|\widehat{X}\left(f_m, f_n\right)\right|^2$ is quantified via non-parametric dispersion measures – i.e. the interquartile range (IQR), the interdecile range (IDR) and the 80 percent quantile (80%). Each statistic is calculated for a particular value of the smoothing parameter M used to average the estimator $F_X(j,k)$ in equation (6).

Given a sampled time series $\{x_1, x_2, ..., x_N\}$ from a stochastic process $\{X_t\}$, the standardized bispectrum is estimated for all possible frequency pairs in the principal domain, obtaining therefore P estimates. This estimation procedure is repeated for all possible values of the smoothing parameter M over the admitted range. Then, for every set of P estimates, each of the three dispersion measures discussed above is calculated.

The discrepancy with respect to the null hypothesis is measured as the difference between the sample dispersion statistic calculated on the estimated standardized bispectrum $2\left|\widehat{X}\left(f_m, f_n\right)\right|^2$ and the same dispersion statistic estimated on the chi-square distribution

$\chi_2^2\left(\widehat{\lambda}_0\right)$ under the null hypothesis of linearity, i.e. constancy of the noncentrality parameter $\lambda_{m,n} = \lambda_0$, where the estimator $\widehat{\lambda}_0$ is given by

$$\widehat{\lambda}_0 = \frac{2\sum_{(m,n)}\left|\widehat{X}\left(f_m, f_n\right)\right|^2}{P} - 2. \tag{11}$$

Considering that the expected value of a non-central chi-square is $E\left[\chi_r^2\left(\lambda\right)\right] = \lambda + r$, where r are the degrees of freedom and λ the non-centrality parameter, it is straightforward to show that $\widehat{\lambda}_0$ is a consistent estimator for $\lambda_{m,n}$, and that the distribution $\chi_2^2\left(\widehat{\lambda}_0\right)$ converges to $\chi_2^2\left(\lambda_0\right)$, when the sample size N, and consequently P increase.

The upper and the lower bounds of the smoothing parameter range are chosen as follows: the highest value of M (M^H) is obtained from the requirement that the number of frequency pairs in the principal domain must exceed one in order to calculate a dispersion measure. The lowest value of M (M^L) is determined by the requirement that the estimator of the non-centrality parameter $\widehat{\lambda}_0$ in (11) must be positive.

The *Maximal Statistics* are respectively

- *Maximal IQR Fractile*

$$MD_{IQR} = \max_{M^L \leq \text{M} \leq M^H}\left\{{}_M D_{IQR}\right\}, \tag{12}$$

where the *Standardized IQR Fractile* is given by

$${}_M D_{IQR} = \frac{\left\{\chi^2_{2,(3)}\left(\lambda_{m,n}\right) - \chi^2_{2,(1)}\left(\lambda_{m,n}\right)\right\} - \left\{\chi^2_{2,(3)}\left(\widehat{\lambda}_0\right) - \chi^2_{2,(1)}\left(\widehat{\lambda}_0\right)\right\}}{\widehat{\sigma}_0}, \tag{13}$$

and where the estimated variance of the IQR fractile sampling distribution $\widehat{\sigma}_0^2$ is given by David (1970) as:

$$\widehat{\sigma}_0^2 = \frac{\left\{3\left[\chi^2_{2,(1)}\left(\widehat{\lambda}_0\right)\right]^{-2} - 2\left[\chi^2_{2,(1)}\left(\widehat{\lambda}_0\right)\chi^2_{2,(3)}\left(\widehat{\lambda}_0\right)\right]^{-1} + 3\left[\chi^2_{2,(3)}\left(\widehat{\lambda}_0\right)\right]^{-2}\right\}}{16P}. \tag{14}$$

Note that the difference in (13) is inherently non-negative since the dispersion measures are positive by construction, and are also significantly "inflated" in case of a nonconstant bispectra, which is what gives the test its power to detect nonlinearity..

- *Maximal IDR Fractile*

$$MD_{IDR} = \max_{M^L \leq \text{M} \leq M^H}\left\{{}_M D_{IDR}\right\}, \tag{15}$$

where

$$ {}_M D_{IDR} = \frac{\left\{\chi^2_{2,(9)}\left(\lambda_{m,n}\right) - \chi^2_{2,(1)}\left(\lambda_{m,n}\right)\right\} - \left\{\chi^2_{2,(9)}\left(\widehat{\lambda}_0\right) - \chi^2_{2,(1)}\left(\widehat{\lambda}_0\right)\right\}}{\widehat{\sigma}_0} \tag{16} $$

is the *Standardized IDR Fractile*, and the variance is given by

$$ \widehat{\sigma}_0^2 = \frac{\left\{3\left[\chi^2_{2,(1)}\left(\widehat{\lambda}_0\right)\right]^{-2} - 2\left[\chi^2_{2,(1)}\left(\widehat{\lambda}_0\right)\chi^2_{2,(9)}\left(\widehat{\lambda}_0\right)\right]^{-1} + 3\left[\chi^2_{2,(9)}\left(\widehat{\lambda}_0\right)\right]^{-2}\right\}}{16P}. \tag{17} $$

- *Maximal 80% Fractile*

$$ MD_{80\%} = \max_{M^L \leq \mathrm{M} \leq M^H} \left\{ {}_M D_{80\%} \right\}, \tag{18} $$

where

$$ {}_M D_{80\%} = \frac{\chi^2_{2,(80)}\left(\lambda_{m,n}\right) - \chi^2_{2,(80)}\left(\widehat{\lambda}_0\right)}{\widehat{\sigma}_0} \tag{19} $$

is the *Standardized 80% Fractile*, and the variance is

$$ \widehat{\sigma}_0^2 = \frac{\{0.8(1-0.8)\}}{\left\{\left[\chi^2_{2,(80)}\left(\widehat{\lambda}_0\right)\right]^2 P\right\}}. \tag{20} $$

The test statistics ${}_M D_{IQR}$, ${}_M D_{IDR}$, and ${}_M D_{80\%}$ are asymptotically standard normals under the null hypothesis that $\{X_t\}$ is a linear process, as defined by (2). However, since there is little reason to believe that these asymptotics converge adequately for empirically reasonable sample lengths, we obtain their sampling distributions in practice using the non-parametric bootstrap.

4 MONTE CARLO SIMULATION RESULTS ON THE NEW BISPECTRAL TEST

4.1 SIZE OF THE NEW BISPECTRAL TESTS

Simulation results on the sizes of the three new bispectral tests are presented in this section using $N_{MC} = 1000$ Monte Carlo repetitions. Recalling that the size of a test is the probability of wrongly rejecting the null hypothesis when this latter is true, three sets of i.i.d observations from a $N(0,1)$, a Student's $t(5)$ and a gamma $G(5,1)$ distribution, respectively, were generated and the rejection significance level for each of the three versions of the maximal test was obtained (using $N_{boot} = 1000$ bootstrap simulations) for each Monte Carlo repetition. Since the null hypothesis of a constant bispectrum is satisfied by construction for these simulated data, the empirical size is estimated as the proportion of rejections of the null hypothesis of linearity out of the N_{MC} Monte Carlo trials. Recalling that the binomial distribution of the empirical size of a test converges to the normal distribution $N\left[\alpha, \alpha\left(1-\alpha\right)/N_{MC}\right]$, as N_{MC} increases, the bootstrapped size of a 5% test should lie within the interval $[0.036; 0.064]$ for $N_{MC} = 1000$.

A sample of $N = 350$ observations was generated from all three probability distributions. For this sample length, the computationally feasible lower limit M^L of the smoothing parameter range is found to be 8 and the upper limit M^H is found to be 48. Table 1 reports the estimated size of the three maximal tests for serially i.i.d. data generated from each of these three distributions:

Table 1: *Size of the Maximal Statistics*

Maximal Statistics			
	MD_{IQR}	MD_{IDR}	$MD_{80\%}$
$N\left(0,1\right)$	0.054	0.043	0.047
$t(5)$	0.042	0.044	0.036
$G\left(5,1\right)$	0.055	0.050	0.057

It is evident that all of the sizes of the bootstrapped maximal tests lie within the confidence interval. Both the Student's t and the gamma distribution are fat-tailed, with kurtosis statistics equal to 9 and 3.2, respectively. And the gamma distribution $G\left(5,1\right)$ has a positive asymmetry statistic equal to 4.5. Thus, all of the maximal Hinich tests appear to be correctly sized at this sample length, even in the presence of kurtosis or skewness in the data.

4.2 POWER OF THE MAXIMAL TEST STATISTICS

Simulation results on the power of the three new bispectral tests are presented in this section, again using $N_{MC} = 1000$ Monte Carlo repetitions. Recalling that the power of a test is the probability of correctly rejecting the null hypothesis when this latter is false, data sets from a variety of different nonlinear stochastic processes were artificially generated. A high value of the estimated power means that the test is particularly sensitive with respect to the form of nonlinear dependence characterizing the stochastic process under question.

Several well-known parametric models nonlinear in the conditional mean, conditional variance, and both, were considered in generating the data sets. The main features of these models strongly depend on the parameter values. We have here chosen the parameter values most often encountered in the literature on nonlinear testing (Barnett et al., 1997; Lee, White and Granger, 1993; Ashley and Patterson, 2000). The generating models considered are

1. **NLMA(2)**
   $X_t = \varepsilon_{t-1} + 0.8\varepsilon_{t-1}\varepsilon_{t-2}$.
2. **Bilinear**
   $X_t = 0.7\varepsilon_{t-1}X_{t-2} + \varepsilon_t$.
3. **ARCH(4)**
   $X_t = h_t^{0.5}\varepsilon_t$
   $h_t = 0.000019 + 0.846\left[0.28X_{t-1}^2 + 0.08X_{t-2}^2 + 0.02X_{t-3}^2 + 0.01X_{t-4}^2\right]$.
4. **GARCH(1,1)**
   $X_t = h_t^{0.5}\varepsilon_t$
   $h_t = 0.0108 + 0.1244X_{t-1}^2 + 0.8516h_{t-1}$.
5. **TAR(2,1)**
   $X_t = -0.5X_{t-1} + \varepsilon_t$ if $X_{t-1} \leq 1$
   $X_t = +0.4X_{t-1} + \varepsilon_t$ otherwise.
6. **Two State Markov(2,1)**
   $X_t = -0.5X_{t-1} + \varepsilon_t$ if in state 1
   $X_t = +0.4X_{t-1} + \varepsilon_t$ if in state 2
   (remain in state with probability $p = 0.90$).
7. **EAR(2,1)**
   $X_t = \left[0.9 + 0.1e^{-X_{t-1}^2}\right]X_{t-1} - \left[0.2 + 0.1e^{-X_{t-1}^2}\right]X_{t-2} + \varepsilon_t$.
8. **Rational NLAR**
   $X_t = (0.7\,|X_{t-1}|)\,/\,(|X_{t-1}| + 2) + \varepsilon_t$.
9. **Exponential Damped AR(2)**
   $X_t = e^{-0.1X_{t-1}^2}\left[0.5X_{t-1}X_{t-2}\right] + \varepsilon_t$.
10. **Logistic(4) Map**
   $X_t = 4X_{t-1}(1 - X_{t-1})$.

where the initial condition X_0 is randomly drawn from a $U(0,1)$ distribution.

In all cases, the innovation series $\{\varepsilon_t\}$ follows a $NIID(0,1)$ stochastic process.

In Table 2 the estimated power is given for both the classical and the new maximal versions of the Hinich bispectral test for linearity. With a sample length of 350 observations, the value of the smoothing parameter M for the classical Hinich bispectral test is the integer part of $350^{0.6}$, which equals 34. As with the size estimates reported above, the computationally feasible lower and upper limits for M used in the maximal test are $M^L = 8$ and $M^H = 48$, respectively, for this sample length. As for the size estimates, 1000 bootstap repetitions were used in characterizing the sampling distribution for each test and 1000 MonteCarlo samples were generated from each generating process listed above.

Table 2: *Power of the bispectral tests for linearity*

	Classical Hinich Test			**Maximal Test**		
	$IQR\ fr.$	$IDR\ fr.$	$80\%\ fr.$	MD_{IQR}	MD_{IDR}	$MD_{80\%}$
$NLMA$	0.124	0.140	0.137	0.311	0.439	0.391
BL	0.110	0.125	0.105	0.234	0.359	0.292
$ARCH$	0.158	0.141	0.134	0.173	0.209	0.195
$GARCH$	0.194	0.196	0.211	0.554	0.697	0.629
TAR	0.231	0.215	0.232	0.363	0.446	0.449
TSM	0.060	0.066	0.063	0.060	0.072	0.056
EAR	0.038	0.047	0.045	0.028	0.037	0.020
$RatNLAR$	0.053	0.050	0.055	0.033	0.037	0.036
$ExpDAR$	0.266	0.267	0.270	0.426	0.471	0.456
$Logistic$	0.272	0.280	0.244	0.872	0.920	0.944

The new Maximal Hinich tests generally show a substantial improvement in the power compared to that of the corresponding classical Hinich tests. It is also notable that the maximal test based on the interdecile range MD_{IDR} appears to be usually more powerful than the other two maximal tests. In particular, the largest power improvements over the classical Hinich tests are with data generated by the *NLMA*, *GARCH* and *Logistic Map* nonlinear alternatives, which are about 215%, 260% and 270%, respectively.

Note that the classic Hinich bispectral test is known to have low power against the forms of deteministic chaos which have typically been analyzed. In this context, it is notable that the maximal bispectral tests proposed here have quite high power to detect the nonlinear serial dependence in data generated from the logistic map.

5 THE INVARIANCE PROPERTY

The property of invariance to linear filtering for polyspectra of any order (Brillinger, 1975) is here considered in order to evaluate the bispectral tests behavior with respect to mis-specified linear pre-filtering models.

Let $\{X_t\}$ be a discrete strictly stationary stochastic process, and $\{Y_t\}$ defined as $Y_t = \sum_{j=0}^{\infty} \psi_j X_{t-j}$, where the coefficients sequence $\{\psi_j\}$ satisfies the summability condition $\sum_{j=0}^{\infty} \psi_j^2 < \infty$. The kth-order polyspectrum of $\{Y_t\}$ is given by

$$\begin{aligned} g_Y(f_1, f_2, ..., f_{k-1}) &= \Psi(f_1)\Psi(f_2) \times ... \times \Psi(f_{k-1}) \times \\ &\times \sum_{\tau_1=-\infty}^{+\infty} \sum_{\tau_2=-\infty}^{+\infty} ... \sum_{\tau_{k-1}=-\infty}^{+\infty} cum(\tau_1, \tau_2, ..., \tau_{k-1}) \times \\ &\times \exp\left\{-i2\pi \sum_{j=1}^{k-1} \tau_j f_j\right\} \\ &= \Psi(f_1)\Psi(f_2) \times ... \times \Psi(f_{k-1})\, g_X(f_1, f_2, ..., f_{k-1}). \end{aligned} \tag{21}$$

where $cum(\tau_1, \tau_2, ..., \tau_{k-1})$ is the joint cumulant of the set $\{X_t, X_{t+\tau_1}, ..., X_{t+\tau_{k-1}}\}$ and $\Psi(f)$ is the transfer function of the linear filter. It follows from (21) that the bispectrum of the stochastic process $\{Y_t\}$ can be written as

$$B_Y(f_1, f_2) = \Psi(f_1)\Psi(f_2)\Psi^*(f_1 + f_2)\, B_X(f_1, f_2), \tag{22}$$

where $B_X(f_1, f_2)$ is defined as (1).

Recalling that the spectrum of the linear process $\{Y_t\}$ can be written as

$$S_Y(f) = |\Psi(f)|^2 S_X(f), \tag{23}$$

the equivalence therorem (Ashley, Hinich and Patterson, 1986) demonstrates that the squared skewness function of $\{Y_t\}$ is given by

$$\begin{aligned}\Gamma_Y^2(f_1, f_2) &= \frac{|B_Y(f_1,f_2)|^2}{S_Y(f_1)\,S_Y(f_2)\,S_Y(f_1+f_2)} \\ &= \frac{|\Psi(f_1)\,\Psi(f_2)\,\Psi^*(f_1+f_2)\,B_X(f_1,f_2)|^2}{\left[|\Psi(f_1)|^2 S_X(f_1)\right]\left[|\Psi(f_2)|^2 S_X(f_2)\right]\left[|\Psi^*(f_1+f_2)|^2 S_X(f_1+f_2)\right]} \\ &= \frac{|B_X(f_1,f_2)|^2}{S_X(f_1)\,S_X(f_2)\,S_X(f_1+f_2)} = \Gamma_X^2(f_1, f_2),\end{aligned} \tag{24}$$

hence $\{X_t\}$ and $\{Y_t\}$ have identical squared skewness functions. Consequently, the bispectral linearity test statistic is asymptotically invariant to linear filtering of the data.

Since the bispectral linear tests (classical or maximal) are thus invariant to linear filtering, these tests can in principle be applied directly to the original data series. In contrast, all other linearity tests require pre-whitening because they are generally quite sensitive to linear as well as nonlinear serial dependence. In practice one must pre-whiten the data for the Hinich tests as well, so as to estimate the test statistics' sampling distributions using the bootstrap, but this invariance property implies that the nonlinearity test results from a bispectral test will be relatively insensitive to errors in specifying and estimating the pre-whitening models.

For example, we generated 350 observations from an autoregressive model of order 2:

$$X_t = 0.4X_{t-1} + 0.3X_{t-2} + \varepsilon_t$$

in which the innovations, the sequence $\{\varepsilon_t\}$, were independently and identically generated from several different probability distributions. The bispectral tests were then applied to the estimated residuals from a pre-whitening model mis-specified to be an $AR(1)$ process. Table 3 below gives the estimated size of the various bispectral tests in each case:

Table 3: *Size of the bispectral linearity tests using mis-specified pre-whitening model*

	Classical Hinich Test			**Maximal Test**		
	$IQR\ fr.$	$IDR\ fr.$	$80\%\ fr.$	MD_{IQR}	MD_{IDR}	$MD_{80\%}$
$N(0,1)$	0.051	0.059	0.057	0.048	0.055	0.044
$t(5)$	0.042	0.036	0.038	0.053	0.060	0.051
$G(5,1)$	**0.065**	**0.071**	**0.068**	0.053	0.062	0.057

Note that the estimated sizes of all three maximal tests lie within the 95% confidence interval of $[0.036; 0.064]$ around the nominal size of 5% either for normally or for asymmetrically distributed data from the Student's $t(5)$ and the gamma $G(5,1)$ distribution[1]. The maximal tests appear to be an improvement on the classical Hinich tests in this regard at this sample length.

6 CONCLUSION

In this paper we have introduced a new bispectral test for nonlinear serial dependance based on a modification of the existing Hinich bispectral test. Notably, the new modified test eliminates the need for a user to select a value for the smoothing parameter, M, as is required in applying the classic Hinich test. The new bispectral test also shows a substantial improvement in its power with respect to a large class of nonlinear stochastic processes. All bispectral tests have the signal advantage of being relatively robust to errors in specifying the order of an $AR(p)$ pre-whitening model; the new maximal tests appear to be even more robust in this regard than the classical Hinich bispectral test.

[1] In contrast, the actual size of the 5% BDS test (Brock, Dechert and Scheinkman, 1987) in this case — unit normal distribution innovations and embedding dimension m=2 — is 0.178; this test is evidently far more sensitive to mis-specification in the order of the pre-whitening model than are the maximal Hinich tests.

7 REFERENCES

[1] Ashley, R. A. and Patterson, D. M. (1989). Linear versus nonlinear macroeconomics: a statistical test. *Int. Econ. Rev.*, **30**, 685-704.

[2] Ashley, R. A. and Patterson, D. M. (2000). *Nonlinear Time Series Workshop. A Toolkit for Detecting and Identifying Nonlinear Serial Dependence*. Kluwer Academic Publishers.

[3] Ashley, R. A., Patterson, D. and Hinich, M. (1986). A diagnostic test for nonlinear serial dependence in time series fitting errors. *J. Time Ser. Anal.*, **7**, 169-178.

[4] Barnett, W. A., Gallant, A. R., Hinich, M. J., Jungeilges, J., Kaplan, D. and Jensen, M. J. (1997). A single-blind controlled competition among tests for nonlinearity and chaos. *J. Econom.*, **82**, 157-192.

[5] Brillinger, D. R. (1965). An introduction to polyspectra. *Ann. Math. Statist.*, **36**, 1351-1374.

[6] Brillinger, D. R. (1975). *Time Series. Data Analysis and Theory.* Holt, Rinehart and Winston, Inc.

[7] Brillinger, D. R. and Rosenblatt, M. (1967). Asymptotic theory of k-th order spectra. In *Spectral Analysis of Time Series*, (ed. B. Harris), pp. 153-188. Wiley, New York.

[8] Brillinger, D. R. and Rosenblatt, M. (1967). Computation and interpretation of k-th order spectra. In *Spectral Analysis of Time Series*, (ed. B. Harris), pp. 189-232. Wiley, New York.

[9] Brock, W. A., Dechert W. D. and Scheinkman, J. A. (1987). A test for independence based on the correlation dimension. SSRI Working Paper 8702, Department of Economics, University of Wisconsin at Madison.

[10] Brock, W. A., Hsieh, D. A. and LeBaron, B. (1991). *Nonlinear Dynamics, Chaos, and Instability: Statistical Theory and Economic Evidence*. MIT Press, Reading, Ma.

[11] David, H. A. (1970). *Order Statistics.* John Wiley and Sons, Inc.

[12] De Lima, P. J. F. (1996). Nonlinearities and Nonstationarities in Stock Returns. Economic Working Paper 360, Department of Economics, The John Hopkins University at Baltimore.

[13] Granger, C. W. J. and Hatanaka, M. (1964). *Spectral Analysis of Economic Time Series*. Princeton University Press. New Jersey.

[14] Granger, C. W. J. and Teräsvirta, T. (1993). *Modelling Nonlinear Economic Relationships*. Oxford University Press, New York.

[15] Hasselman, K., Munk, W. and MacDonald, G. (1963). Bispectra of ocean waves. In *Time Series Analysis*, (ed. Rosenblatt). John Wiley, New York.

[16] Helland, K., Lii, K. and Rosenblatt, M. (1979). Bispectra and energy transfer in grid-generated turbulence. In *Developments in Statistics*, (ed. Krishnaiah). Academic Press, New York.

[17] Hinich, M. J. (1982). Testing for Gaussianity and linearity of a stationary time series. *J. Time Ser. Anal.*, **3**, 169-175.

[18] Huber, P. J., Kleiner, B. and Gasser, T. (1971). Statistical methods for investigating phase relations. *IEEE Trans. Aud. Electr.*, **19**, 78-86.

[19] Kim, Y. and Powers, E. (1978). Digital bispectral analysis of self-excited fluctuation spectra. *Phys. Fluids*, **21**, 1452-1453.

[20] Lee, T. H., White, H. and Granger, C. W. J. (1993). Testing for neglected nonlinearity in time series models. *J. Econom.*, **56**, 269-290.

[21] Lemos, M. and Stokes, H. H. (2000). A single-blind controlled competition among tests for nonlinearity and chaos; further results. Unpublished Memo, UIC Department of Economics.

[22] Lii, K., Rosenblatt, M. and Van Atta, C. (1976). Bispectral measurements in turbulence. *J. Fluid Mech*., **77**, 45-62.

[23] Rusticelli, E. (2005). A sequential and iterative testing procedure to identify the generating structure of time series and a new bispectral test for nonlinearity. PhD Thesis, Department of Statistics, University of Bologna.

[24] Sato, T., Sasaki, K. and Nakamura, Y. (1977). Real-time bispectral analysis of gear noise and its application to contractless diagnosis. *J. Acoust. Soc. Amer.*, **62**, 382-387.

[25] Shaman, P. (1965). Large-sample approximations to the first and second-order moments of bispectral estimates. Technical Report, New York University, School of Engineering and Science.

[26] Stokes, H. H. (1991, 2nd ed. 1997). *Specifying and Diagnostically Testing Econometric Models.* Quorum Books, New York.

[27] Subba Rao, T. and Gabr, M. (1980). A test for linearity of stationary time series analysis. *J. Time Ser. Anal.*, **1**, 145-158.

[28] Tong, H. (1990). *Non-linear Time Series*. Clarendon Press Oxford.

[29] Tukey, J. W. (1959). An introduction to the measurement of spectra. In *Probability and Statistics*, (ed. U. Grenander), pp. 300-330. Wiley, New York.

[30] Van Atta, C. (1979). Inertial range bispectra in turbolence. *Phys. Fluids*, **22**, 1440-1442.